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TESTING THE LONG-RUN RISK MODEL: A KALMAN FILTER APPROACH

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P 13 D 2018

T L -R R B (2004) K f M L . O f f - f f . I , f - M L , L -R R 70 f US . H , f f f .

Ke o d : E ; - ; K f .

1. Introduction

D , - -f .[1] M P (1985) f , . ., ff f - - f f 6% -f . A , (**RRA**) (> 50) 6% U.S . S , -f ,

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[‡]C .
[1]P C (2017) f - -f .

-f W (1989). T , f
f - S D F (**SDF**)
, . .,
H -J (H J , 1991) B ,
(> 250).
M f M P (1985)
f ff . S E Z
(1989), C C (1999), B (2004), B (2006),
. E Z (1989) f f -
f f , - ,
f f - (EIS)
. W E Ƶ (E Z , 1989, 1991) -
f , -
6%
-f , EIS E Ƶ
f . T , E Ƶ f -f , . .,
-f ,
.

B (2004) - (LRR)
. T f LRR
. F f E Ƶ f ,
f -
. T f ,
(“L -
R ”). W , . ., f -
1.5 10, LRR
f US . H , -
. F , f -
. I
f AR(1) (
AR(1) ff $\rho = 0.98$) (
σ). S , f
. I ,
- f f -
f . S
- f ,
f -
K f . T - f
f f f

. W M -H
ﬁ ﬁ ML .
T K ﬁ ﬁ ,
ﬁ . F , B
N (1999) K ﬁ
G ﬁ . S S
(2000) K ﬁ -ﬁ ﬁ
- - -
ﬁ . P H (1990) ﬁ
ﬁ K ﬁ .

μ_c f , σ_c, σ f
f - . T -
ε_t ε_t^c f *i.i.d.*
f . T f $_t$
f .[2] N f f
B (2004). W
B (2004) ,
,
.

$$_t = \frac{\beta}{1-\beta}\left(\mu_c + (1-\gamma)\left(\frac{\sigma_c^2}{2} + \frac{\beta^2}{2(1-\beta\rho)^2}\sigma^2\right)\right) + c_t + \frac{\beta}{1-\beta\rho}\ _t. \tag{6}$$

O SDF M_{t+1}

$$M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\left[\frac{[(1-\gamma)\ _{t+1}]}{E_t[\ ((1-\gamma)\ _{t+1})]}\right], \tag{7}$$

f c_{t+1} $_{t+1}$ f :

$$\begin{aligned} _{t+1} = & \ \beta - (\mu_c + \ _t) - \frac{1}{2}(1-\gamma)^2\left(\sigma_c^2 + \left(\frac{\beta\sigma}{1-\beta\rho}\right)^2\right) \\ & + (1-\gamma)\frac{\beta\sigma}{1-\beta\rho}\varepsilon_{t+1} - \gamma\sigma_c\varepsilon_{t+1}^c. \end{aligned} \tag{8}$$

I [3]

$$E(M_{t+1}) = \beta \quad \left(-\mu_c + \left(\gamma - \frac{1}{2}\right)\sigma_c^2 + \frac{\sigma^2}{2(1-\rho^2)}\right), \tag{9}$$

$$\sigma_{M_{t+1}} = E(M_{t+1}) \cdot \sqrt{\quad \left(\frac{\sigma^2}{1-\rho^2} + (\gamma\sigma_c)^2 + \left(\frac{(1-\gamma)\beta\sigma}{1-\beta\rho}\right)^2\right) - 1}. \tag{10}$$

3. Data and Model Performance

T . W US
- ,
S&P 500 3- T , . f F
R E D (FRED) . T R f , -

6 F -F f . A f 1948**Q2** 2011**Q3**.

G , H -J f SDF ( ) f f , ( ) ( ) 6 F -F P f (F F , 1989). A F . 1,

f SDF. SDF f .

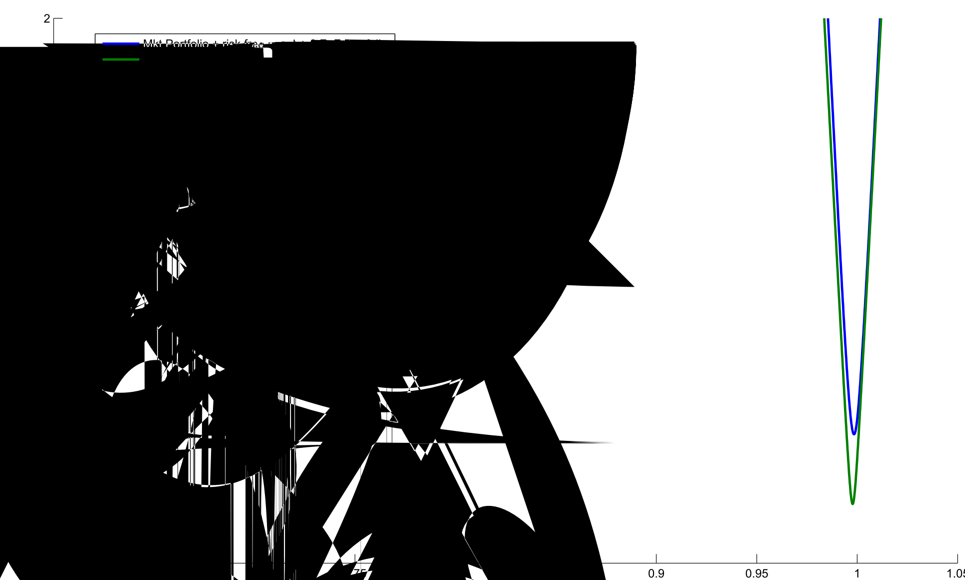
Case I: CRRA. If CRRA f , SDF f f

$$M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}. \tag{11}$$

B C_t f f

$$c_{t+1} = c_t + \mu_c + \sigma_c \varepsilon^c_{t+1}, \quad \varepsilon^c_{t+1} \sim i.i.d.\ \mathrm{N} \quad (0, 1), \tag{12}$$

$$E(M_{t+1}) = \beta \quad \left[-\gamma\mu_c + \frac{1}{2}(\gamma\sigma_c)^2\right], \qquad \mathrm{V} \quad (M_{t+1}) = E(M_{t+1}) \cdot \sqrt{\quad (\gamma^2\sigma_c^2) - 1}. \tag{13}$$

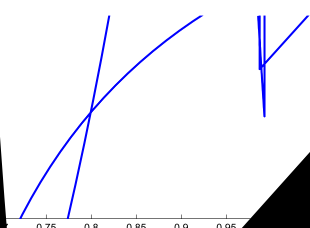


T 1. P .

	M	
V	P ( ) CRRA E Z	P ( ) L -
β	0.995	0.995
μ_c	0.0042	0.0042
σ_c	0.0062	0.0062
ρ	—	0.98
σ	—	0.00062

N e: T f . T f β 0.995 f . F CRRA f E Z f , μ_c σ_c f E . (12) P ( ). P ( ) f L - . W f B (2004) f ρ σ .

μ_c σ_c^2 f , [redacted] f M_{t+1} f ff [redacted] I CRRA f . I [redacted] - f H –J B [redacted] γ 220. A γ [redacted] (f



[redacted]

H J (H J , 1997)), , RRA M P (1985). T γ EIS RRA f CRRA f . D , γ 6% .

Case II: Epstein–Zin. I E Z f $\eta = 1$, SDF [4]

$$E(M_{t+1}) = \beta \left[-\mu_c + \sigma_c^2 \left(\gamma - \frac{1}{2} \right) \right], \qquad \sigma_{M_{t+1}} = E(M_{t+1}) \cdot \sqrt{ (\gamma^2 \sigma_c^2) - 1}. \tag{14}$$

S f , μ_c σ_c^2 f T 1 P ( ), f M_{t+1} f ff γ' F . 2( ). S f E Z f , EIS RRA η γ , $\gamma \simeq 85$ f . A , RRA γ .

Case III: Long-Run risk. T - , t , f (f). T T 1, P ( ). T f μ_c σ_c^2 P ( ). F ρ σ^2, f B (2004). A B , t, (39% f f B (2004)). I , $\gamma \simeq 30$ f , F . 2( ).

4. Estimation of the Long-Run Risk Component

A t ( ) f , . B (2004) t f AR(1) f ρ 0.98. M , - , $\sigma \simeq \frac{\sigma_c}{10}$. I ,

[4] S A B f .

ρ σ

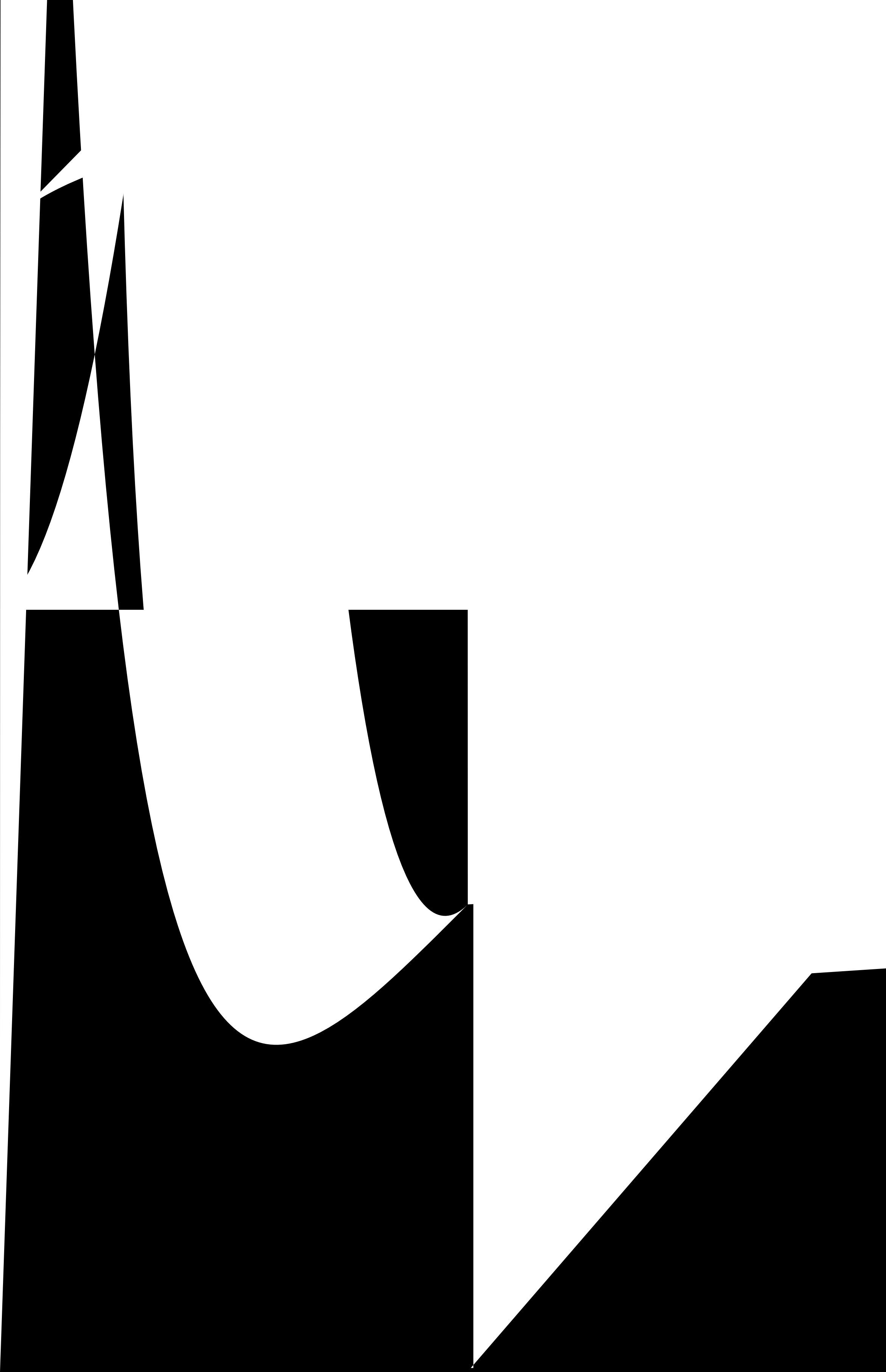
T 2. T MLE f LRR ff .

N	M	E
μ_c	0.004274	0.004239	0.003894
σ_c	0.004628	0.004999	0.005185
ρ	0.741086	0.90	0.98
σ	0.002719	0.001826	0.001422

N e: T f
LRR ff f . T f
. T

ff $\rho \in [0.9, 1)$. I , f
$\rho \in [0.98, 1)$.

MLE f f f K F
E . (18). T
f f T 2. I
$\hat{\mu}_c$ 0.004215,
f f ff f
. I E . (18), $\sigma \simeq \frac{\sigma_c}{10}$ MLE
$\hat{\sigma} \simeq \frac{\hat{\sigma}_c}{2}$. T B ’ -
$\hat{\sigma}_c$ $\hat{\sigma}$ f f .
T - f ff f
. O $\hat{\rho} = 0.741$ f -
, f f f
E . (5), f B
(2004) .
W f (**PDF**) f
M –H . I
ML . W f 50,000
PDF f f F . 3. T
f μ_c PDF f σ
σ_c, . A -
MLE, μ_c, σ_c, σ f -
. I f f ρ. T f
f f f MLE f
(0.72, 0.8). B , f ρ
f . T f ’
B (2004) ’
ρ 0.98 (
0.9). W f σ_c σ



f P (B) f F . 2, f f
f LRR .

5. Conclusion

I , K F M L
f - B (2004).
W ρ
f LRR . T f -
ρ. U , -
f σ_c σ f M –H f
$\sigma = \sigma_c$,
. A , , LRR
70 f f US . M
f -
B (2004) . H ,
f .

Acknowledgments

T C P S F (G
N . 2018M641302).

Appendix A. Value Function

G E . (3) f $_t = k_0 + k_1 c_t + k_2 \,_t$, k_0, k_1
k_2 . F f f
E . (3),

$$_t = (1-\beta)c_t + \frac{\beta}{1-\gamma} \quad [E_t(\quad ((1-\gamma)(k_0 + k_1 c_{t+1} + k_2 \,_{t+1})))]. \tag{A.1}$$

S E . (4) (5) E . (A.1)

$$\begin{aligned} _t &= (1-\beta)c_t + \frac{\beta}{1-\gamma} \quad [\quad ((1-\gamma)(k_0 + k_1(c_t + \mu_c) + (k_1 + \rho k_2) \,_t))] \\ &\quad + \frac{\beta}{1-\gamma} \quad E_t[\quad ((1-\gamma)(k_1\sigma_c\varepsilon^c_{t+1} + k_2\sigma \,\varepsilon_{t+1}))] \\ &= \beta\left(k_0 + k_1\mu_c + (1-\gamma)\left(\frac{k_1^2\sigma_c^2 + k_2^2\sigma^2}{2}\right)\right) + (1-\beta+\beta k_1)c_t \\ &\quad + \beta(k_1 + k_2\rho) \,_t. \end{aligned} \tag{A.2}$$

T f . N LHS f
E . (A.2) $k_0 + k_1 c_t + k_2 \,_t$ f . B f-
f f E . (A.2),

$$
\begin{aligned}
k_0 &= \frac{\beta}{1-\beta}\left(\mu_c + (1-\gamma)\left(\frac{\sigma_c^2}{2} + \frac{\beta^2}{2(1-\beta\rho)^2}\sigma^2\right)\right),\\
k_1 &= 1,\\
k_2 &= \frac{\beta}{1-\beta\rho}.
\end{aligned}
$$

Appendix B. SDF

If f f E Ƶ f EIS $= \eta$ $a = \gamma$,
$\mu_{t+1} = (E_t U_{t+1}^{1-\gamma})^{\frac{1}{1-\gamma}}$, SDF

$$M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta} \cdot \left(\frac{U_{t+1}}{\mu_{t+1}}\right)^{\eta-\gamma} \tag{B.1}$$

F $\eta = 1$,

$$M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1} \cdot \left(\frac{[(1-\gamma)\ _{t+1}]}{E_t[\ \ ((1-\gamma)\ _{t+1})]}\right) \tag{B.2}$$

T ,

$$_{t+1} = \quad \beta - \Delta c_{t+1} + (1-\gamma)\ _{t+1} - \quad E_t[\quad ((1-\gamma)\ _{t+1})]. \tag{B.3}$$

S E . (6), (4) (5) E . (B.4),

$$
\begin{aligned}
_{t+1} = &\quad \beta - (\mu_c + \ _t) - \frac{1}{2}(1-\gamma)^2\left(\sigma_c^2 + \left(\frac{\beta\sigma}{1-\beta\rho}\right)^2\right)\\
&+ (1-\gamma)\frac{\beta\sigma}{1-\beta\rho}\varepsilon_{t+1} - \gamma\sigma_c\varepsilon_{t+1}^c.
\end{aligned} \tag{B.4}
$$

B f $\{\varepsilon_t^c, \varepsilon_t\}$, $_{t+1}$ f
M_{t+1} f - . T f ,
f M_{t+1}

$$E(M_{t+1}) = \beta \qquad \left(-\mu_c + \left(\gamma - \frac{1}{2}\right)\sigma_c^2 + \frac{\sigma^2}{2(1-\rho^2)}\right)$$

$$\sigma_{M_{t+1}} = E(M_{t+1}) \cdot \sqrt{\qquad \left(\frac{\sigma^2}{1-\rho^2} + (\gamma\sigma_c)^2 + \left(\frac{(1-\gamma)\beta\sigma}{1-\beta\rho}\right)^2\right) - 1}.$$

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